Stable solitons of even and odd parities supported by competing nonlocal nonlinearities

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We introduce a one-dimensional phenomenological model of a nonlocal medium featuring focusing cubic and defocusing quintic nonlocal optical nonlinearities. By means of numerical methods, we find families of solitons of two types, even-parity (fundamental) and dipole-mode (odd-parity) ones. Stability of the solitons is explored by means of computation of eigenvalues associated with modes of small perturbations, and tested in direct simulations. We find that the stability of the fundamental solitons strictly follows the Vakhitov-Kolokolov criterion, whereas the dipole solitons can be destabilized through a Hamiltonian-Hopf bifurcation. The solitons of both types may be stable in the nonlocal model with only quintic self-attractive nonlinearity, in contrast with the instability of all solitons in the local version of the quintic model.

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I. INTRODUCTION

Optical solitons, i.e., localized waves maintaining their profile in nonlinear optical media due to the balance between the group-velocity dispersion, diffraction, and nonlinear self-phase modulation, have been in the focus of experimental and theoretical studies in the last decade [1–5]. However, solitons in media with the cubic ($\chi^{(3)}$) self-focusing nonlinearity, obeying the cubic nonlinear Schrödinger (NLS) equation, are unstable in two and three dimensions (2D and 3D), due to the possibility of collapse (blowup of the wave packet) in the same dimension [6]. In fact, the (critical) collapse occurs in the one-dimensional (1D) NLS equation with the self-focusing quintic term too [7].

On the other hand, the collapse does not occur in multidimensional media with the nonlocal $\chi^{(3)}$ nonlinearity [8,9], therefore such media may give rise to *stable* solitons, see, e.g., a recent comprehensive review in Ref. [10]. 2D spatial solitons stabilized by nonlocal nonlinearities were observed in vapors [11] and lead glasses featuring strong thermal nonlinearity [12]; in the latter case, elliptic and vortex-ring solitons were reported. Optical 1D solitons supported by a nonlocal $\chi^{(3)}$ nonlinearity were also created in liquid crystals [13,14]. Further, such issues as self-focusing in photorefractive media [15], periodic lattices [16], vortices [17], spatial solitons in soft matter [18], attraction between dark solitons [19], dependence of the stability domain of 2D solitons on the spatial profile of the nonlocal-response function [20], and 3D spatiotemporal solitons [21], were all considered in the context of nonlocality.

Another natural physical setting supporting stable multidimensional solitons is provided by *competing nonlinearities*, such as ones represented by combinations of cubic-quintic (CQ) [22–24] and quadratic-cubic types [25–27]. In most cases [22,23,27–29], the competing nonlinearities were considered in the context of the stabilization of *spinning solitons* (alias vortex rings or vortex tori, in the 2D and 3D cases, respectively): while fundamental (zero-vorticity) multidimensional solitons are stable in quadratic media without the addition of cubic nonlinearity, their spinning counterparts are unstable in the same case [30] (see also Ref. [31]). Competing cubic and quintic nonlinearities stabilize 2D [32] and 3D [33] dissipative spinning solitons too, which has been demonstrated in the framework of the complex CQ Ginzburg-Landau equation. In addition, studied in some detail was also the stabilization by dint of competing nonlinearities of necklace-shaped soliton clusters carrying angular momentum [34]. In the framework of discrete models, it has been demonstrated that the CQ nonlinearity supports a great variety of stable solitons (including asymmetric ones) in the 1D lattice model [35].

The aim of this work is to study the effect of competition between self-focusing cubic and self-defocusing quintic *nonlocal nonlinearities* on the existence and stability of even-parity (fundamental) solitons and their dipole-mode (odd-parity) counterparts. While the solitons of the former type are supported by local nonlinearities as well, the odd-parity ones may only exist in the presence of a nonlocal nonlinear response of the medium.

The paper is organized as follows: after introducing a general phenomenological model describing nonlocal media with the competing CQ nonlinearities in Sec. II, we present basic numerical results that demonstrate the existence of even- and odd-parity soliton families in Sec. III. In the same section, their stability borders are accurately delineated. Direct numerical simulations of the evolution of perturbed solutions are shown to be in full accordance with the calculations of instability eigenvalues. The paper is concluded by Sec. IV.

II. THE MODEL

The 1D phenomenological model of a nonlocal medium with competing CQ nonlinearities is based on the following scaled equation for the amplitude q of the electromagnetic field propagating along coordinate ξ and diffracting in the transverse direction η,

$$iq_{\xi} + (1/2)q_{\eta\eta} + qn = 0, \tag{1}$$

where the local perturbation of the refractive index, n, is determined by the integral expressions

$$n = \alpha_3 \int_{-\infty}^{+\infty} G_3(\eta - \zeta)|q(\zeta,\xi)|^2 d\zeta + \alpha_5 \int_{-\infty}^{+\infty} G_5(\eta - \zeta)|q(\zeta,\xi)|^4 d\zeta. \tag{2}$$

Here the Green functions $G_{3,5}$ account for the response functions of the nonlinear material. Typically, the response functions are either exponential ones (as in liquid crystals), or Gaussians [10]. Below, we assume the latter, by setting

$$G_{3,5}(\eta - \zeta) = (\pi d_{3,5})^{-1/2}\exp[-(\eta - \zeta)^2/d_{3,5}], \tag{3}$$

where the coefficients d_3 and d_5 determine the corresponding nonlocality ranges of the cubic and quintic nonlinearities, while the coefficients in front of the Gaussians follow from the normalization conditions, $\int_{-\infty}^{+\infty} G_{3,5}(\eta)d\eta = 1$. The strength of the cubic nonlinearity may be scaled to set $\alpha_3 = 1$ in Eq. (2), and, as mentioned above, we consider the case of the competition between the cubic and quintic nonlinearities, that is, $\alpha_5 < 0$ (implying the saturation of the nonlinear response). A dynamical invariant of the system (1) is the beam power,

$$U = \int_{-\infty}^{+\infty} |q|^2 d\eta. \tag{4}$$

For $d_3 = d_5 = 0$, which corresponds to the local CQ medium, with G_3 and G_5 being δ functions, Eqs. (2) and (3) yield $n = \alpha_3|q|^2 + \alpha_5|q|^4$, hence Eq. (1) reduces to the usual local CQ model,

$$iq_{\xi} + (1/2)q_{\eta\eta} + \alpha_3|q|^2 q + \alpha_5|q|^4 q = 0. \tag{5}$$

A family of stable exact soliton solutions to Eq. (5) with an arbitrary wave number b, is well known for $\alpha_5 < 0$ [36],

$$q(\eta,\xi) = 2\sqrt{\frac{b}{\alpha_3 v(\eta)}}\exp(ib\xi),$$

$$v(\eta) = 1 + \sqrt{1 + \frac{16\alpha_5 b}{3\alpha_3^2}}\cosh(2\sqrt{2b}\eta), \tag{6}$$

provided that $1 + (16\alpha_5 b)/(3\alpha_3^2) > 0$. Recently, this family was also extended to the case of $\alpha_5 > 0$ [37]; in that case, the family is stable too, despite the above-mentioned possibility of the collapse in the 1D model with $\alpha_5 > 0$.

In the particular case of $\alpha_3 = 0$, the quintic NLS equation, with $\alpha_5 = 1$, yields another known exact solution,

$$q_Q(\eta,\xi) = \frac{(3b)^{1/4}}{\sqrt{\cosh(2\sqrt{2b}\eta)}}\exp(ib\xi), \tag{7}$$

which is unstable, unlike solitons (6). The integral power of solution (7) does not depend on b, $U_Q = (\pi/2)\sqrt{3/2} \approx 1.924$, which is explained by the fact that this soliton plays the role of the *separatrix* between collapsing solutions, with $U > U_Q$, and decaying ones, with $U < U_Q$ [6,7].

III. NUMERICAL SOLUTIONS FOR EVEN- AND ODD-PARITY SOLITONS AND THEIR STABILITY ANALYSIS

We look for stationary solutions of Eqs. (1) and (2) in the form $q(\eta,\xi) = w(\eta)\exp(ib\xi)$, where the real function $w(\eta)$ obeys the following equations:

$$w_{\eta\eta} + 2wn - 2bw = 0, \tag{8}$$

$$n(\eta) = \alpha_3 \int_{-\infty}^{+\infty} G_3(\eta - \zeta)w^2(\zeta)d\zeta + \alpha_5 \int_{-\infty}^{+\infty} G_5(\eta - \zeta)w^4(\zeta)d\zeta. \tag{9}$$

We have numerically found families of localized solutions to these equations, treating the corresponding two-point boundary-value problem with the standard band-matrix algorithm.

To analyze the linear stability of solitons, we have searched for perturbed solutions to Eqs. (1) and (2) as

$$q(\eta,\xi) = \{w(\eta) + [u(\eta) + iv(\eta)]e^{\lambda\xi}\}\exp(ib\xi), \tag{10}$$

where a perturbation eigenmode with real and imaginary parts $u(\eta,\xi)$ and $v(\eta,\xi)$ can grow with a complex rate, $\lambda = \mathrm{Re}(\lambda) + i\,\mathrm{Im}(\lambda)$. Linearization of the equations around a stationary solution yields a system of equations from which λ can be found numerically.

In Figs. 1(a) and 1(b) we display the power U vs propagation constant b for families of even-parity (fundamental) solitons for the model with both equal and unequal nonlocality ranges, d_3 and d_5, of the competing cubic and quintic nonlinearities. For numerical calculations, we have fixed the strengths of the nonlinear terms in Eq. (2) to be $\alpha_3 = 1$, and $\alpha_5 = -0.2$. We see that, as in the case of local media for which $d_3 = d_5 = 0$ [see the lower curve in Fig. 1(a)], the power U increases monotonically with the propagation constant b. Thus, the fundamental solitons in the nonlocal CQ media may be stable, as they satisfy the *Vakhitov-Kolokolov (VK) criterion*, $dU/db > 0$, which is a necessary (but, generally, not sufficient) stability condition for the fundamental soliton family in equations of the NLS type [6,38]. Through numerical computation of the instability eigenvalues for this family, we have checked that the VK criterion is actually a sufficient condition for the stability for the fundamental solitons.

In Fig. 1(c), we display the dependence $U = U(b)$ for the nonlocal medium with the purely quintic nonlinearity ($\alpha_3 = 0$, $\alpha_5 = 1$). The particular case of unstable solitons (7) in the local quintic medium is represented by the lower curve in Fig. 1(c); as said above, in this case the total power $U(b) = U_Q \approx 1.924$ does not depend on the wave number b, and, accordingly, they are marginally stable solutions in terms of the VK stability criterion. In fact, as is common to separatrix solutions in the case of the critical collapse [6,39], solitons (7) are subject to a subexponential instability, that is why the

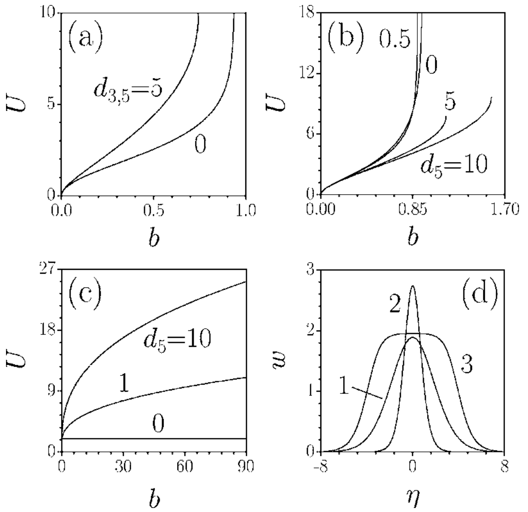

FIG. 1. The power U, see Eq. (4), versus propagation constant b for even-parity (fundamental) soliton families with different nonlocality ranges in the nonlocal cubic-quintic model, (a) and (b), and in the nonlocal quintic model (c). In (a) and (b), $\alpha_3=1$ and $\alpha_5=-0.2$, whereas in (c) $\alpha_3=0$ and $\alpha_5=+1$. (d) Typical stationary field profiles for propagation constants near the maximum value $b_{\max}$. The other parameters are: $d_3=d_5=10$ (curve 1), $d_3=1$, $d_5=10$ (curve 2), and $d_3=1$, $d_5=0.5$ (curve 3). The same values, $\alpha_3=1$, $\alpha_5=-0.2$, and $\alpha_3=0$, $\alpha_5=+1$, are adopted in all other examples of solitons in the cubic-quintic and pure quintic model, respectively, which are displayed below.

VK criterion predicts a zero critical eigenvalue for them. However, in the nonlocal model with the purely quintic nonlinearity the fundamental solitons are *stable*, in precise agreement with the prediction of the VK criterion [see Fig. 1(c)], which has been corroborated by numerical computation of the full set of the stability eigenvalues for this soliton family.

We see from Figs. 1(a) and 1(b) that, in the case of the nonlocal CQ medium, the lower (cutoff) value of the propagation constant is $b=0$, and the power U increases monotonically until the propagation constant reaches a maximum value $b_{\max}$. However, in the case of the nonlocal quintic medium ($\alpha_3=0$), $b_{\max}=\infty$, see Fig. 1(c). Typical stationary soliton's profiles for values of the propagation constant near the maximum value $b_{\max}$ are shown in Fig. 1(d), for several sets of values of d_3 and d_5. We see that, depending on the specific values of nonlocality parameters d_3 and d_5, the field profile may be well localized or, on the contrary, it may display a flat-top profile (recall that, in the case of the local CQ medium, the soliton's shape is a flat-top-like one for values of the propagation constants near the maximum value).

In Fig. 2(a) we plot the total power U vs propagation constant b for odd-parity (dipole-mode) solitons, in the nonlocal CQ medium with $d_3=d_5=10$. We see that the $U=U(b)$ dependence is, generally, not monotonous, displaying a characteristic loop; accordingly, the propagation constant, b, is bounded to a certain interval of variation, between a minimum (cutoff) value, which is different from zero, and a maximum one. In a certain interval of variation of b, three different solutions are found for a given value of the propagation constant, which correspond to different soliton's profiles, and different values of total power U. We found also that while the dependence $U=U(b)$ remains qualitatively similar when we decrease the nonlocality range of the cubic nonlinearity d_3, the domain of existence for dipole solitons quickly shrinks. Thus, already for $d_3=8$ at $d_5=10$ the domain of existence shrinks to $0.6174<b<0.727\,03$. Stability analysis reveals that dipole solitons are stable on the entire lower branch of curve $U=U(b)$ and even on a small part of upper branch of $U=U(b)$ curve corresponding to negative value of dU/db. Thus, two different stable dipole solitons having equal propagation constants can coexist in a certain interval of variation of the propagation constant b in the model that we consider. Notice, that the black curves in Figs. 2(a) and 2(b) denote stable soliton branches, whereas the red ones depict unstable soliton branches. The interval of stability for the dipole solitons in Fig. 2(a) corresponding to $d_3=d_5=10$ is given by $0.4496<b<0.6750$, while when we decrease the nonlocality range of the cubic nonlinearity from $d_3=10$ to $d_3=8$ the stability interval shrinks to $0.6174<b<0.7245$.

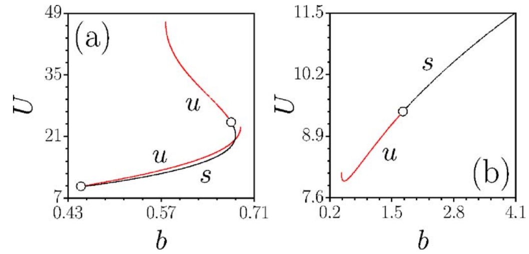

FIG. 2. (Color online) Power U vs propagation constant b for the odd-parity (dipole-mode) solitons in the nonlocal cubic-quintic model with $d_3=d_5=10$ (a) and in the quintic nonlocal model, with $d_5=10$ (b). Circles indicate the transition points between stable (marked by label s) and unstable (marked by label u) soliton branches. Stable branches are shown by black lines, whereas unstable ones are shown by red (dark gray) lines.

In Fig. 2(b) we plot the total power U vs propagation constant b for the dipole-mode (odd-parity) solitons in the nonlocal quintic model with $d_5=10$. Here, too, stable and unstable solution branches are displayed as black and red lines, respectively. The cutoff value of b is $b_{\rm co}=0.440\,783$, and the critical point separating stable and unstable solitons is located at $b_{\rm cr}=1.7338$. Note that, in the quintic model, all characteristic values of b scale with the nonlocality parameter d_5, as $1/d_5$. We stress again that the very fact of the existence of stable solitons (this time, of the dipole type) in the model with a purely quintic self-focusing nonlinearity is remarkable, as in the local limit this model gives rise solely to unstable solitons.

In Fig. 3(a), we display typical profiles of the stationary odd-parity solitons for the nonlocal CQ model with $d_3=d_5=10$. Here, as before, the black lines (curve 2) denote the stable solitons, and the red ones (dark gray, in the black-and-white version) stand for unstable ones (curves 1 and 3). Typical profiles of the odd-parity solitons in the nonlocal quintic model with $d_5=10$ are displayed in Fig. 3(b). We see that the two humps of the unstable dipole solitons are much farther separated in comparison to their stable counterparts. This

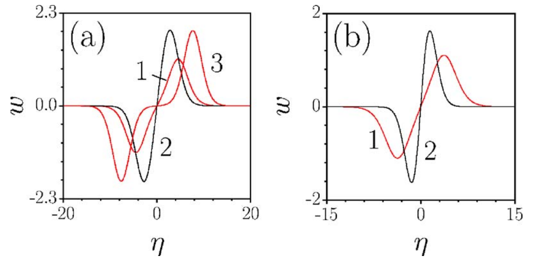

FIG. 3. (Color online) Typical profiles of odd-parity solitons in the nonlocal cubic-quintic medium with $d_3=d_5=10$ (a) and in the quintic medium, with $d_5=10$ (b). In panel (a), $b=0.449\,37$ (curve 1), $b=0.681\,62$ (curve 2), and $b=0.689\,18$ (curve 3). In panel (b), $b=0.4407$ (curve 1) and $b=1.75$ (curve 2).

feature results in two generic instability scenarios: the two humps of the unstable soliton either fuse into a stable fundamental (even-parity) soliton, or they split into two out-of-phase stable even-mode solitons (see below).

In Fig. 4, by red (dark gray) and black curves we plot the real and imaginary parts of the critical instability eigenvalues [ones that determine the (in)stability], λ_1 and λ_2, vs propagation constant b of the unperturbed solution for the odd-parity solitons in the nonlocal CQ and quintic models [panels Figs. 4(a)–4(d), respectively]. The parameters are the same as in cases displayed in Fig. 2. Stable are those solitons for which $\mathrm{Re}(\lambda)=0$, see Eq. (10) (only the neutral stability is possible in the conservative systems). The two critical points located at $b_{\mathrm{cr}}=0.4496$ and $b_{\mathrm{cr}}=0.6750$ are clearly seen in Figs. 4(a) and 4(b). Note that Fig. 4(c) shows the relevant instability eigenvalues corresponding to the unstable branch [see the lower red (dark gray) line marked by label u in Fig. 2(a)] that is very close to the lower stable branch in Fig. 2(a) marked by label s.

Though we are dealing with odd-parity solitons, in the case of the nonlocal CQ model we observe essentially the

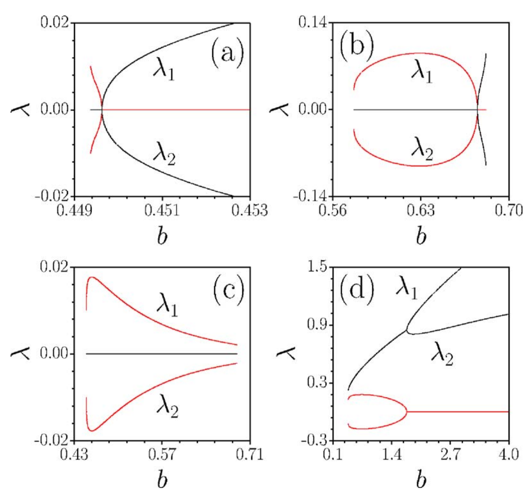

FIG. 4. (Color online) Real and imaginary parts [shown by red (dark gray) and black lines, respectively] of two eigenvalues responsible for the Hamiltonian-Hopf bifurcation of the odd-parity solitons. In (a), (b), and (c), we fix $d_3=d_5=10$, whereas in (d), which pertains to the quintic model, we set $d_5=10$. In fact, panel (d) also implies the presence of the symmetric branches of $\mathrm{Im}(\lambda)$, with the opposite sign.

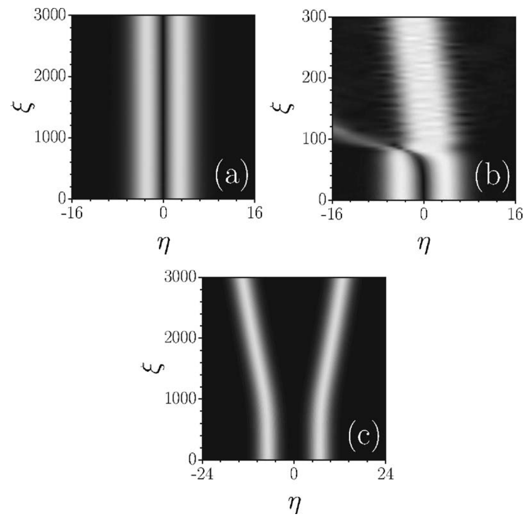

FIG. 5. Simulated evolution of stable (a) and unstable (b) and (c) odd-parity solitons in the presence of white input noise. Here $d_3=d_5=10$, $b=0.6$, and the input power is $U=13.66$ (a), $U=37.13$ (b), and $U=14.55$ (c).

same situation as one familiar for fundamental (even-parity) solitons, where the relevant instability eigenvalues are either real or pure imaginary. However, in the case of the nonlocal quintic model, we get a typical oscillatory instability [40], which occur, for example, in the studies of spinning [23,24] solitons, when the eigenvalues are complex, resulting in the *Hamiltonian-Hopf bifurcation*, see Fig. 4(d). It is relevant to recall that, as said above, we have checked that the stability regions of the fundamental (even-parity) solitons in the nonlocal quintic model are precisely predicted by the VK criterion, the corresponding eigenvalues being real.

Finally, the predictions of the linear stability analysis were checked in direct simulations of Eqs. (1), which were run by means of the standard Crank-Nicholson scheme. First, we have checked that all the even-parity (fundamental) solitons that were predicted above to be stable, are stable indeed against random perturbations. Next, we have explored the evolution of odd (dipole-mode) solitons in the nonlocal CQ model. Figure 5(a) displays an example of the self-healing (relaxation to the unperturbed shape) of a stable low-power odd-parity soliton under white-noise perturbations at the amplitude level of 10% [the parameters are the same as in Fig. 2(a)]. Figures 5(b) and 5(c) show the typical instability scenarios of the dipole solitons with a higher power [which correspond to the two upper branches in Fig. 2(a)]. Thus, depending on the value of their integral power and the distance between the two humps, the unstable odd-parity soliton either fuses into a stable fundamental one [if the initial separation between the two humps is small enough, see Fig. 5(b)], or splits into two distinct out-of-phase stable fundamental solitons, if the initial distance between its humps is large, see Fig. 5(c).

IV. CONCLUSIONS

We have introduced a general one-dimensional nonlocal model with competing self-attractive cubic and self-repulsive quintic nonlinearity. Two types of solitons were constructed in a numerical form, fundamental (even-parity) and dipole-mode (odd-parity) ones, the latter type do not exist in the local limit. The stability analysis, performed through the computation of the growth rate for eigenmodes of small perturbations, and verified in direct simulations, demonstrates that the stability of the fundamental solitons exactly obeys the Vakhitov-Kolokolov criterion. For odd-parity solitons, in the case of the nonlocal cubic-quintic model we have found that the relevant instability eigenvalues are either real or pure imaginary, whereas in the case of the nonlocal quintic model, we got a typical oscillatory instability, when the eigenvalues are complex, resulting in a Hamiltonian-Hopf bifurcation. A noteworthy fact is that the solitons of both types have their stability regions even in the nonlocal model with the purely quintic attractive nonlinearity, in the local version of which all solitons are unstable. The stable states predicted in this work can be realized as spatial solitons in optical media featuring nonlocal nonlinear response.

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